Systems Dynamics

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Outline

- Introduction
- Continuous and discrete dynamics
- Stability of equilibrium points
- Bifurcation analysis of non-linear systems
- Introduction to chaotic behaviour
- Application examples

Introduction

Systems dynamics study the time evolution of process models. Generally, the type of trajectories followed by the states depend on the value of the actions u and the initial conditions x_0, but also on the structure of the mathematical model representing the process and the value of the model parameters p

$$\frac{dx}{dt} = f(x, u, p) \qquad x(0) = x_0$$

Continuous and discrete dynamics

Continuous processes are represented normally by ODEs, DAEs or PDEs involving real variables that change continuously over time taking any value in a given range.

$$\frac{dx}{dt} = f(x,u,p) \qquad x(0) = x_0$$

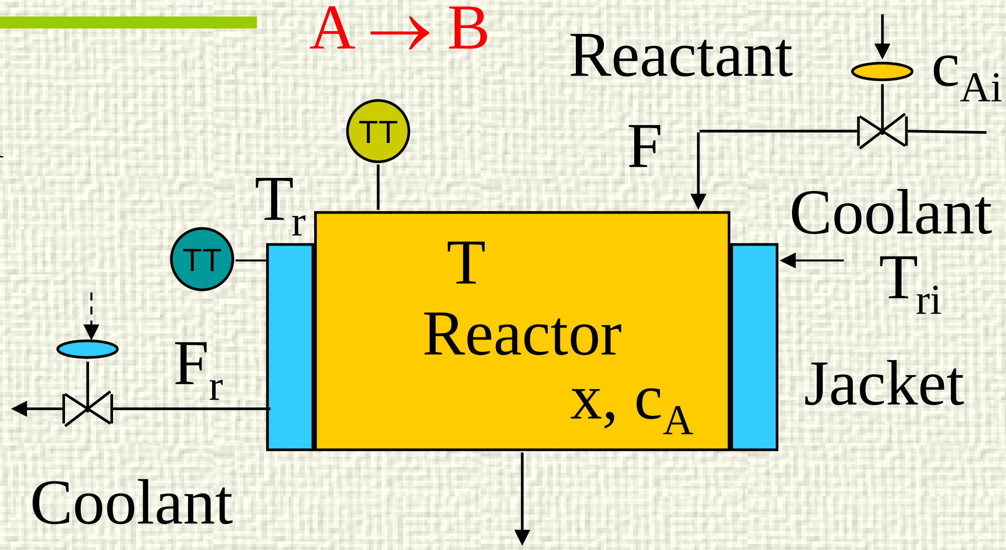
Sampled or discrete systems are represented normally by difference equations involving variables that change only at certain time instants

$$x(k+1) = F(x(k),u(k),p)$$

$$k = kT \quad k = 0,1,2,\ldots$$

$$x(0) = x_0$$

Example: Chemical reactor



$$V\frac{dc_A}{dt} = Fc_{Ai} - Fc_A - Vke^{-E/RT}c_A$$

$$x = (c_{Ai} - c_A)/c_{Ai}$$

Mass balance

Energy balance

$$V\rho c_e \frac{dT}{dt} = F\rho c_e T_i - F\rho c_e T + Vke^{-E/RT}c_A \Delta H - UA(T - T_r)$$

$$V_r \rho_r c_{er} \frac{dT_r}{dt} = F_r \rho_r c_{e_r} T_{ri} - F_r \rho_r c_{er} T_r + UA(T - T_r)$$

Systems dynamics

The local study of systems dynamics, and in particular stability, can be made using the eigenvalues of the linearized model around the considered point.

$$\frac{dx}{dt} = f(x,u,p) \rightarrow \frac{d\Delta x}{dt} = A\Delta x + B\Delta u$$

Points specially important are the equilibrium points.

$$\frac{dx_e}{dt} = f(x_e,u,p) = 0$$

The steady-state points are given by the solution of this set of equations

Systems dynamics

- ✓ The numerical value of the A and B matrices of the linearized models, as well as the equilibrium points, depend on the parameters p

$$\frac{dx}{dt} = f(x,u,p)$$

$$\frac{d\Delta x}{dt} = A(p)_{x_e}\Delta x + B(p)_{x_e}\Delta u$$

$$\left| A(p)_{x_e} - \lambda I \right| = 0$$

λ Eigenvalues of A

If Real(λ_i) > 0 unstable point.
Real negative λ_i creates overdamped dynamics.
Imaginary negative λ_i creates underdamped dynamics.
Real(λ_i) = 0 creates oscillations.

Autonomous systems

For a given input trajectory u(t), the systems dynamics only depends on the initial point x_0. E.g. systems under closed loop control.

$$\frac{dx}{dt} = f(x,u,p) \qquad x(0) = x_0$$

Autonomous system:

$$\frac{dx}{dt} = f(x,p) \qquad x(0) = x_0$$

Dynamics of the autonomous system can be study as a function of the initial point and parameters p

Bifurcations

The numerical value of the A matrix of the linearized model, as well as the equilibrium points, depend on the parameters p

$$\frac{dx}{dt} = f(x,p)$$

$$\frac{d\Delta x}{dt} = A(p)_{x_e} \Delta x$$

Changing p, it may happens that the eigenvalues of A, or the number of equilibrium points, change in such way that the new type of dynamics is created (stable vs. unstable, limit cycle,..). This is called a bifurcation. Then, p is a bifurcation parameter.

$$\left| A(p)_{x_e} - \lambda I \right| = 0$$

λ Eigenvalues of A

Example 1

Possible equilibrium points

$$\frac{dx}{dt} = f(x,\mu) = \mu x - x^3$$

If $\mu \le 0$, there is only one real solution $x_e = 0$

$$f(x_e,\mu) = \mu x_e - x_e^3 = 0$$

If $\mu > 0$, there are three different equilibrium points:

$$x_e(\mu - x_e^2) = 0 \quad \Rightarrow \quad \begin{cases} x_e = 0 \\ \mu = x_e^2 \end{cases}$$

$x_e = 0, \; -\mu^{1/2}, \mu^{1/2}$

The value $\mu = 0$ is a bifurcation point for the system because the number of equilibrium points changes between 1 and 3 at $\mu=0$

This is called a pitchfork bifurcation

Dynamics at the equilibrium points

Example: Non linear system with a parameter μ

$$\frac{dx}{dt} = \mu x - x^3$$

- ✓ Linearized system at equilibrium point x_e

$$\frac{d\Delta x}{dt} = (\mu - 3x_e^2)\Delta x$$

- ✓ Eigenvalues $|A - \lambda I| = 0$

$$\mu - 3x_e^2 - \lambda = 0$$

$$\lambda = \mu - 3x_e^2$$

- ✓ Equilibrium points:

$$f(x_e, \mu) = \mu x_e - x_e^3 = 0$$

$$x_e(\mu - x_e^2) = 0 \quad \Rightarrow \quad \begin{cases} x_e = 0 \\ \mu = x_e^2 \end{cases}$$

Dynamics at the equilibrium points

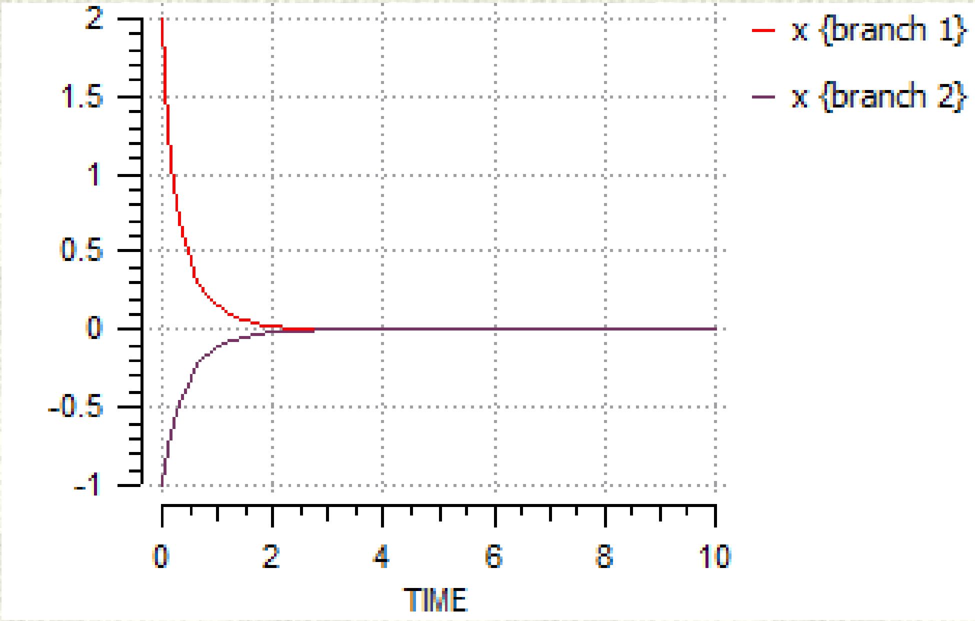
- If $\mu < 0 \;\rightarrow x_e = 0, \quad \lambda < 0$

The origin is a stable overdamped equilibrium point for any initial condition

$$x_e(\mu - x_e^2) = 0 \quad \Rightarrow$$

$$\begin{cases} x_e = 0 \\ \mu = x_e^2 \end{cases}$$

$$\lambda = \mu - 3x_e^2$$



μ = -2
red: x_0 = 2
Black: x_0 = -1

Dynamics at the equilibrium points

- If $\mu > 0 \rightarrow$ three equilibrium points
- $x_e = 0, \quad \lambda = \mu > 0$ unstable point
- $x_e = \mu^{1/2}, \lambda = \mu - 3\mu = -2\mu < 0$ stable overdamped equilibrium
- $x_e = -\mu^{1/2}, \lambda = \mu - 3\mu = -2\mu < 0$ stable overdamped equilibrium

$$x_e(\mu - x_e^2) = 0 \quad \Rightarrow$$

$$\begin{cases} x_e = 0 \\ \mu = x_e^2 \end{cases}$$

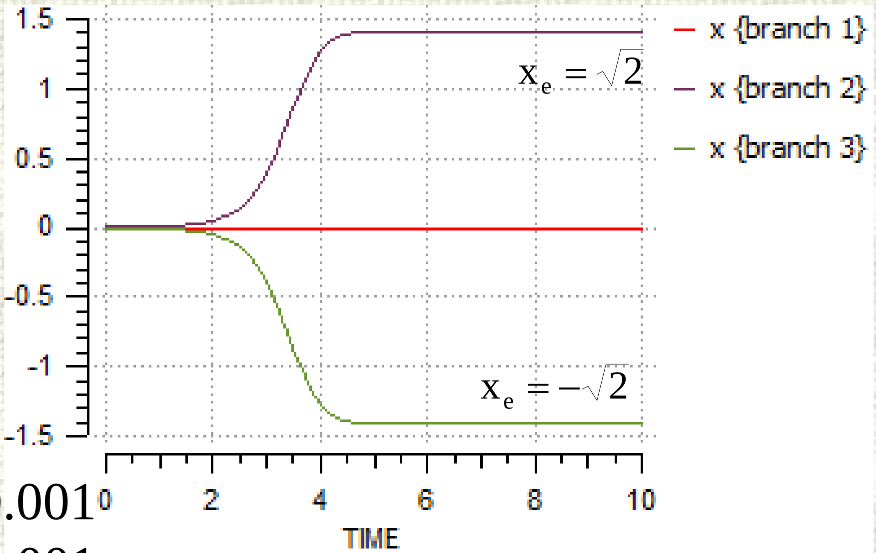
$$\lambda = \mu - 3x_e^2$$

The origin is unstable and each of the two stable overdamped equilibrium points are reached depending on the initial point x_0

$\mu = 2$

purple: $x_0 = 0.001$

green: $x_0 = -0.001$



Example 2

Possible equilibrium points

$$\dot{x}_1 = x_2 + x_1(\mu - x_1^2 - x_2^2)$$
$$\dot{x}_2 = -x_1 + x_2(\mu - x_1^2 - x_2^2)$$

They should satisfy:

$$0 = x_{2e} + x_{1e}(\mu - x_{1e}^2 - x_{2e}^2)$$
$$0 = -x_{1e} + x_{2e}(\mu - x_{1e}^2 - x_{2e}^2)$$

Substituting x_{1e} and x_{2e} in the other equation:

$$0 = x_{2e} + x_{2e}(\mu - x_{1e}^2 - x_{2e}^2)(\mu - x_{1e}^2 - x_{2e}^2)$$
$$x_{2e}(1 + (\mu - x_{1e}^2 - x_{2e}^2)^2) = 0 \quad \Rightarrow x_{2e} = 0$$
$$0 = x_{1e}(1 + (\mu - x_{1e}^2 - x_{2e}^2)^2) \quad \Rightarrow x_{1e} = 0$$

(0,0)' is the only equilibrium point

Stability

(0,0)’ is the only equilibrium point, but notice that a trajectory given by $\mu = x_{1e}^2 + x_{2e}^2$ also satisfies the equilibrium

$$-x_{2e}x_{1e} = x_{1e}^2(\mu - x_{1e}^2 - x_{2e}^2)$$

$$x_{2e}x_{1e} = x_{2e}^2(\mu - x_{1e}^2 - x_{2e}^2)$$

$$0 = (x_{1e}^2 + x_{2e}^2)(\mu - x_{1e}^2 - x_{2e}^2)$$

$$x_{1e} = 0, x_{2e} = 0$$

$$\mu = x_{1e}^2 + x_{2e}^2$$

$$\dot{x}_1 = x_2 + x_1(\mu - x_1^2 - x_2^2)$$

$$\dot{x}_2 = -x_1 + x_2(\mu - x_1^2 - x_2^2)$$

$$\begin{bmatrix} \Delta\dot{x}_1 \\ \Delta\dot{x}_2 \end{bmatrix} = \begin{bmatrix} \mu - 3x_{1e}^2 - x_{2e}^2 & 1 - 2x_{1e}x_{2e} \\ -1 - 2x_{1e}x_{2e} & \mu - x_{1e}^2 - 3x_{2e}^2 \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix}$$

For x_e = (0,0)’

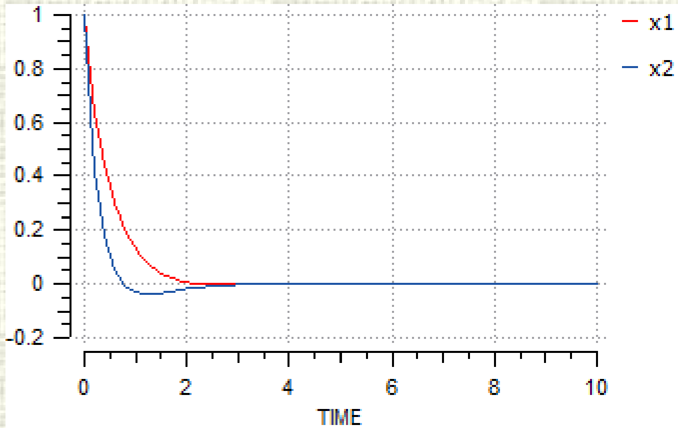
$$A = \begin{bmatrix} \mu & 1 \\ -1 & \mu \end{bmatrix}$$

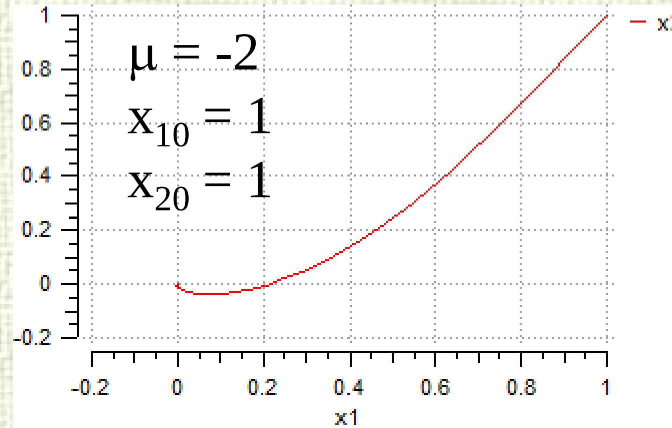
$$\mathrm{eig}(A) = \mu \pm j$$

Stability of x_e

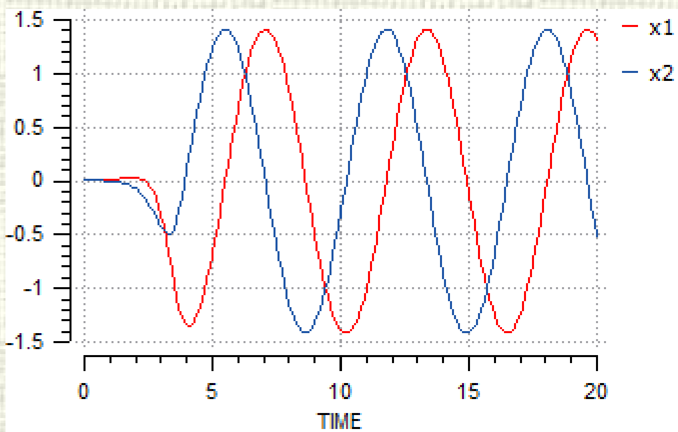
$$eig(A) = \lambda = \mu \pm j$$

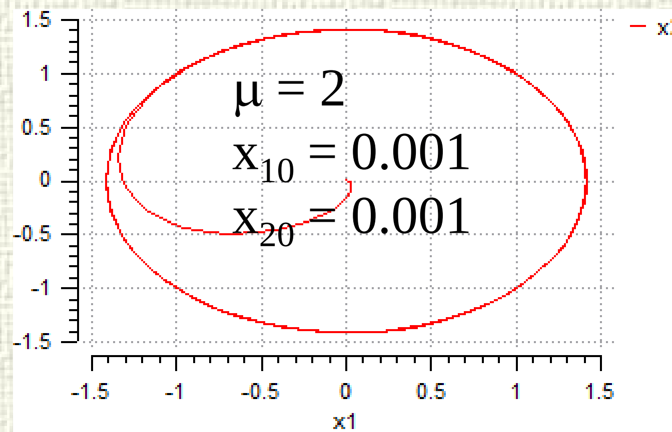
- ✓ If $\mu < 0$, underdamped stable equilibrium point for any initial condition
- ✓ If $\mu > 0$, (0,0)' unstable equilibrium point





Phase plane





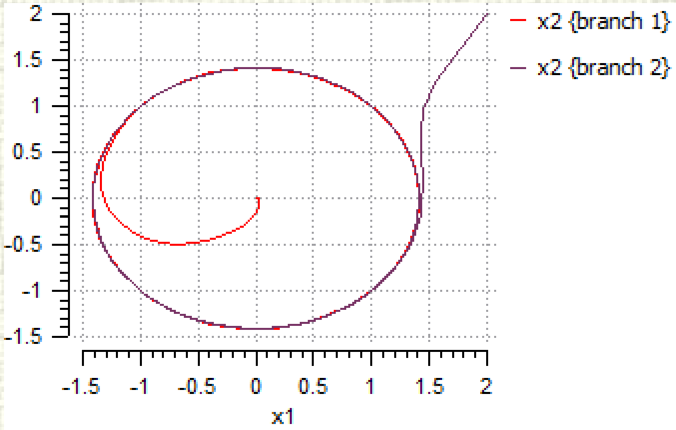
Limit cycle

- If $\mu < 0$, stable equilibrium point for any initial condition
- If $\mu > 0$, unstable equilibrium point
- $\mu = 0$ is a bifurcation point. The system changes dynamics from a stable to unstable equilibrium point and the trajectory moves to a cycle limit. This is called a Hopf bifurcation

For $x_e = (0,0)'$

$$A = \begin{bmatrix} \mu & 1 \\ -1 & \mu \end{bmatrix}$$

$$\text{eig}(A) = \lambda = \mu \pm j$$



Limit cycle: Periodic isolated trajectory

Limit cycle

$$\begin{bmatrix}\Delta\dot{x}_1\\ \Delta\dot{x}_2\end{bmatrix}=\begin{bmatrix}\mu-3x_{1e}^2-x_{2e}^2 & 1-2x_{1e}x_{2e}\\ -1-2x_{1e}x_{2e} & \mu-x_{1e}^2-3x_{2e}^2\end{bmatrix}\begin{bmatrix}\Delta x_1\\ \Delta x_2\end{bmatrix}$$

For points in the trajectory satisfying

$$\mu = x_{1e}^2 + x_{2e}^2$$

$$A=\begin{bmatrix}-2x_{1e}^2 & 1-2x_{1e}x_{2e}\\ -1-2x_{1e}x_{2e} & -2x_{2e}^2\end{bmatrix}$$

$$\begin{vmatrix}-2x_{1e}^2-\lambda & 1-2x_{1e}x_{2e}\\ -1-2x_{1e}x_{2e} & -2x_{2e}^2-\lambda\end{vmatrix}=0$$

$$(-2x_{1e}^2-\lambda)(-2x_{2e}^2-\lambda)-4x_{1e}^2x_{2e}^2-1=0$$

$$\lambda^2+2\lambda(x_{2e}^2+x_{1e}^2)+1=0$$

$$\lambda^2+2\lambda\mu+1=0$$

$$\lambda=-\mu\pm\sqrt{\mu^2-1}\quad \mathrm{Real}(\lambda)<0$$

Solutions are always stable on this trajectory

Stable cycle limit

Chaotic behaviour

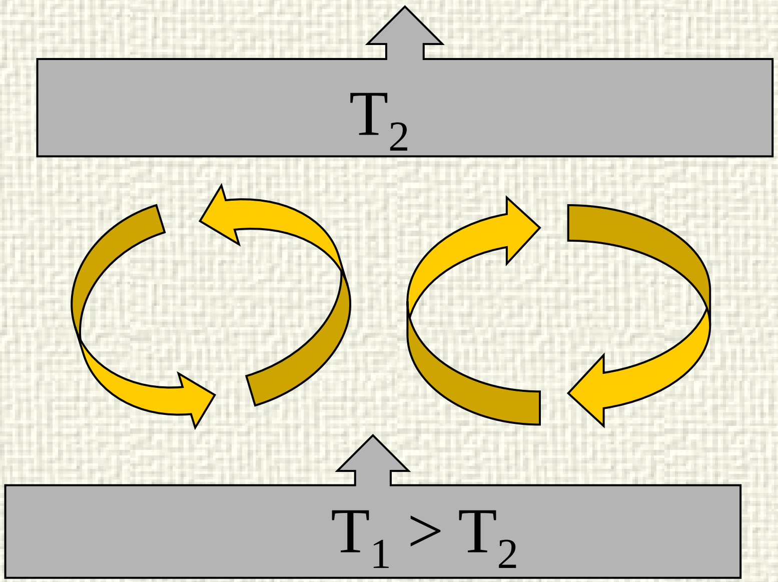
Generally, it is possible to predict the future behaviour of model states as a function of its initial value.

$$\frac{dx}{dt} = f(x,u,p) \qquad x(0) = x_0$$

Nevertheless, certain systems have such huge sensibility to the initial conditions, that it is impossible to predict its long term trajectory. This is called a chaotic behaviour.

If there are no stable equilibrium points and possible cycle limits are unstable, the solution may wander never repeating trajectory and showing a chaotic behaviour

Lorenz equations



Convention rolls due to a temperature difference in a fluid which density decreases with temperature

$$\dot{x}_1 = \sigma(x_2 - x_1)$$
$$\dot{x}_2 = r x_1 - x_2 - x_1 x_3$$
$$\dot{x}_3 = -b x_3 + x_1 x_2$$

x_1 turning speed of the convective rolls

x_2 temperature difference between ascending and descending currents

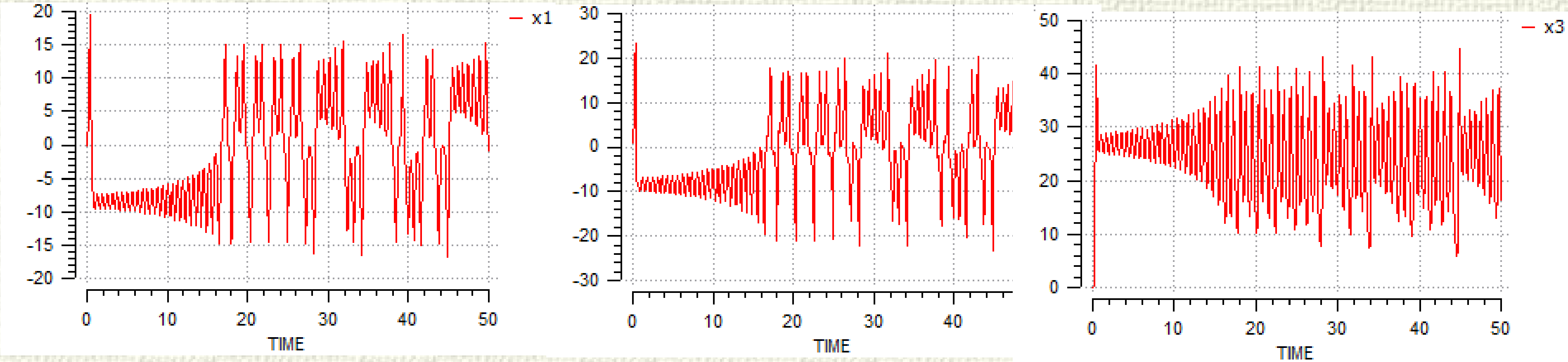
x_3 distortion of vertical temperature profile from linearity

σ Prandtl number

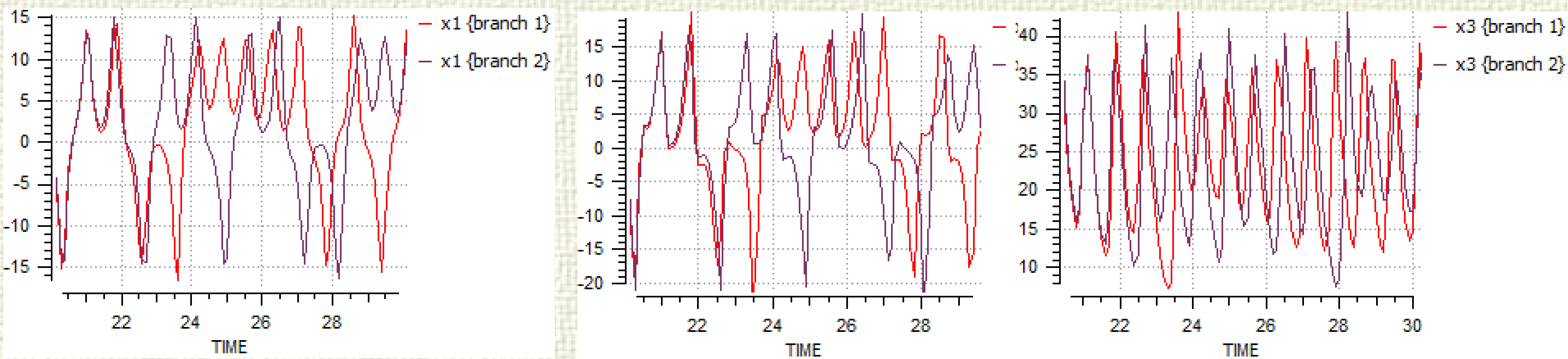
r Rayleigh number/ critical Rayleigh number

b geometric factor

Lorenz equations

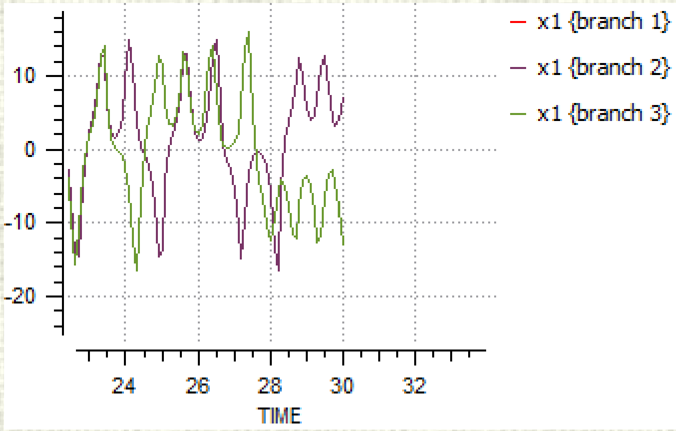


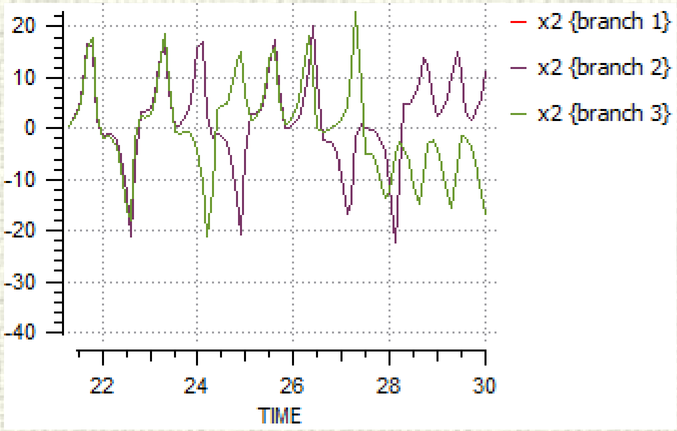
Solution for x_0 = (0,1,0) σ = 10, b = 8/3, r = 28

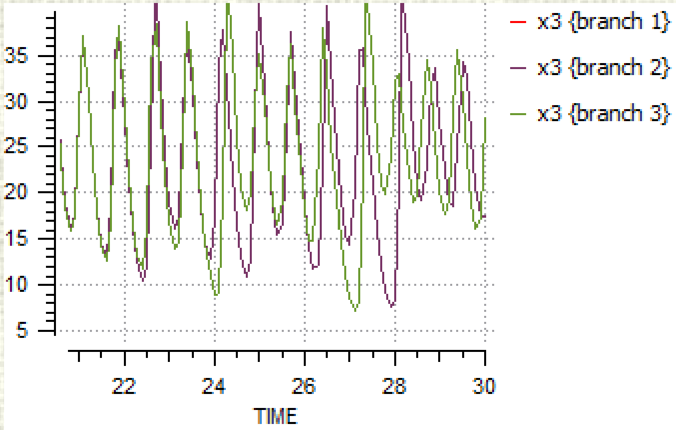


Comparison with the solution for x_0 = (0, 1.01, 0) σ = 10, b = 8/3, r = 28

Lorenz equations







Comparison between the solution for $x_0 = (0, 1, 0)$ $\sigma = 10$, $r = 28$, $b=2.6666667$ and $x_0 = (0, 1, 0)$ $\sigma = 10$, $r = 28$, $b=2.6666666667$